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Supports of Measure Solutions for Spatially Homogeneous Boltzmann Equations

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We prove that the support of the unique measure solution for the spatially homogeneous Boltzmann equation in $\mathbb{R}^3$ is the whole space, if the initial distribution is not a Dirac measure and has 4-order moment. More precisely, we obtain the lower bound of exponential type for the probability of any small ball in $\mathbb{R}^3$ relative to the measure solution.

KEY WORDS: Boltzmann equation, measure solution, support.

1. INTRODUCTION AND MAIN RESULT

Let us consider the spatially homogeneous Boltzmann equation in $\mathbb{R}^3$

$$\frac{\partial}{\partial t} f(\xi, t) = Q(f, f)(\xi, t), \quad f(\xi, 0) = f_0(\xi), \tag{1}$$

where $f(\xi, t)$ is a non-negative function which describes the velocity distribution of the particles in a dilute gas, and $Q(f, f)$ is the Boltzmann collision operator which is given by

$$Q(f, f)(\xi) = \int_{\mathbb{R}^3} \int_{S^2} \mathcal{B}(\xi - \xi_*, \omega)[f(\xi')f(\xi'_*) - f(\xi)f(\xi_*)]d\xi_* d\omega.$$

In the above definition, S^2 is the unit sphere in $\mathbb{R}^3$, and ξ', ξ'_* are the velocities of a pair of particles after a collision, which have velocities ξ, ξ_* before the collision. They are related by

$$\begin{cases} \xi' = \xi - (\xi - \xi_*, \omega)\omega, \\ \xi'_* = \xi_* + (\xi - \xi_*, \omega)\omega. \end{cases} \tag{2}$$

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The non-negative function $\mathcal{B}(\xi - \xi_*, \omega)$ is the so called collision kernel. General theory for the Boltzmann equation can be found in the monographes.[5,7,8]

In this paper, we deal only with the so-called Maxwellian molecules and hard potentials with angular cut-off (including hard sphere models). So, the collision kernel has the form

$$\mathcal{B}(\xi - \xi_*, \omega) := |\xi - \xi_*|^\beta b(\theta), \qquad \theta := \arccos(|\xi_* - \xi|^{-1}|\langle \xi_* - \xi, \omega\rangle|),$$

where $\beta \in [0, 1]$ and

$$\int_0^{\pi/2} b(\theta)\sin\theta d\theta < +\infty \Leftrightarrow K_b := \int_{S^2} b(\theta)d\omega < +\infty. \tag{3}$$

The mathematical theory for the space homogeneous Boltzmann equation, including the existence and uniqueness, moments estimates, pointwise lower bounds and L^p-estimates, is by now rather complete. The readers may find these materials in Ref. 1–4, 9–12, 14, 15, 18–21. In the situation of angular cut-off, the existence and uniqueness was established recently by Mischler-Wennberg[15] under minimal assumption on the initial datum $f_0 \geq 0$:

$$\int_{\mathbb{R}^3} f_0(\xi)(1 + |\xi|^2)d\xi < +\infty.$$

We are concerned with the lower bound estimates in the present paper. This problem can be traced back to Carleman's pioneering work,[4] in which he showed for hard potentials that the positively radial solution to Eq. (1) in weighted L^∞ space:

$$L_6^\infty(\mathbb{R}^3) = \{f(\xi) : \ f(\xi) \text{ is measurable and } \sup_{\xi\in\mathbb{R}^3}(1 + |\xi|^6)|f(\xi)| < \infty\}$$

has an exponential lower bound of the following form

$$\forall t_0 > 0, \varepsilon > 0, \exists K_0, K_1 \Rightarrow f(t, \xi) \geq K_0 e^{-K_1|\xi|^{2+\varepsilon}}, \text{ for all} t \geq t_0, \xi \in \mathbb{R}^3. \tag{4}$$

This result was greatly improved by Pulvirenti-Wennberg in Ref. 19, in which they proved, under the assumption that the initial datum has finite mass, energy and entropy, that a Maxwellian lower bound for the solution to Eq. (1) exists. More precisely, they obtained that

$$\forall t_0 > 0, \exists K_0, K_1 \Rightarrow f(t, \xi) \geq K_0 e^{-K_1|\xi|^2}, \text{ for all } t \geq t_0, \ \xi \in \mathbb{R}^3.$$

In particular, this means that the particles immediately fill up the whole velocity space. In their proof, the Carleman representation plays a crucial role.

It is well known that an exponential type lower bound estimate like (4) for the solution to Eq. (1) plays an important role in the study of the "entropy-entropy production" method to prove the convergence of solution to equilibrium.[6,10,22,23] Recently, this kind of estimates are derived for the spatially inhomogeneous

Boltzmann equation with periodic boundary conditions by Mouhot.[16] Obviously, Pulvirenti-Wennberg's result is a special case of Mouhot's result, furthermore, Mouhot gave a method which enables one to remove the assumption of boundedness of the entropy of initial datum.

We now turn to the spatially homogeneous Boltzmann equation with measure form which was studied recently by the authors in Ref. 24. For $p \geq 0$, let $\mathcal{M}_p$ (resp. $\mathcal{P}_p$) denote the set of finite signed measures (resp. probability measures) on $\mathbb{R}^3$ with finite p-th order moments. Then $\mathcal{M}_p$ is a Banach space under the norm

$$\|\mu\|_{\mathrm{var},p} := \int_{\mathbb{R}^3} (1+|\xi|^2)^{\frac{p}{2}} |\mu|(d\xi),$$

where $|\mu|$ denotes the total variation measure of signed measure μ.

Consider the following spatially homogeneous Boltzmann equation in $\mathcal{M}_2$:

$$\frac{\partial \mu_t}{\partial t} = Q(\mu_t), \qquad \mu_0 = \nu \in \mathcal{P}_4, \tag{5}$$

where the collision operator Q acting on $\mu \in \mathcal{M}_2$ is defined by the bounded linear functional on $C_b(\mathbb{R}^3)$, the set of bounded continuous functions on $\mathbb{R}^3$

$$\begin{aligned} Q(\mu)(\phi) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} &[\phi(\xi') + \phi(\xi_*') - \phi(\xi) - \phi(\xi_*)] \\ &\times \mathcal{B}(\xi - \xi_*, \omega)\mu(d\xi)\mu(d\xi_*)d\omega. \end{aligned} \tag{6}$$

The existence and uniqueness of conservative solution to Eq. (5) were established in Ref. 24. A natural question for the unique solution μ_t is now put forward: for any positive time t, does the distribution μ_t of velocity fill up the whole space? That is to say, would the support of measure μ_t be $\mathbb{R}^3$?

If $\mu_0 = \delta_v$ for some $v \in \mathbb{R}^3$ is a Dirac measure, then the unique solution is given by $\mu_t \equiv \delta_v$. Clearly, this can be explained in physics that no collision occurs if there is only one particle in a gas. Except for this case, we can still ask this question. To solve this problem, a key step is to make a detailed analysis for the transformation (12).[2] Here the Carleman representation is absent. In the present paper we shall give an affirmative answer for this problem and mainly prove that

Theorem 1.1. *Let $B_\varepsilon(\xi_0)$ be the open ball in $\mathbb{R}^3$ with center ξ_0 and radius ε. For fixed $\mu_0 \in \mathcal{P}_4$, assume that there are two distinct points $v, w \in \mathbb{R}^3$ such that for $\varepsilon > 0$ sufficiently small*

$$f(\varepsilon) := \mu_0(B_\varepsilon(w))\mu_0(B_\varepsilon(v)) > 0,$$

then for any $\delta > 0$, there are positive constants $\varepsilon_0 < 1$ and $C_i, i = 1, \cdots, 5$ such that

$$\begin{aligned} \mu_t(B_\varepsilon(\xi_0)) \geq C_1 \exp\{&-C_2 t - C_3|\xi_0|^{2+\delta} + C_4|\xi_0|^2 \\ &\times [\log(\varepsilon f(C_5\varepsilon/(|\xi_0|^\alpha + 1))) - t + \log(t \wedge 1)]\} \end{aligned}$$

for all $\xi_0 \in \mathbb{R}^3$, $\varepsilon \in (0, \varepsilon_0)$ *and* $t > 0$, *where* $\alpha = 2\log 5/\log 2$.

Remark 1.2. Assume that $\mu_0 = \delta_v + \delta_w + \nu$ for two distinct points $v, w \in \mathbb{R}^3$ and $\nu \in \mathcal{P}_4$, then for any $t > 0$ and $\delta > 0$, there are positive constants $\varepsilon_0 < 1$ and $C_t > 0$ such that

$$\mu_t(B_\varepsilon(\xi_0)) \geq C_t \cdot \exp\{C_t|\xi_0|^{2+\delta} \log \varepsilon\}$$

for all $\xi_0 \in \mathbb{R}^3$ and $\varepsilon \in (0, \varepsilon_0]$.

Corollary 1.3. *If* $\mu_0 \in \mathcal{P}_4$ *is not a Dirac measure, then the support of* μ_t *for any* $t > 0$ *is the whole space.*

Proof: We only need to prove that there exist two points v, w such that for any open set $V \ni v$ and $W \ni w$, $\mu_0(V) > 0$ and $\mu_0(W) > 0$. In fact, first, since μ_0 is a probability measure, there is a $v \in \mathbb{R}^3$ such that $\mu_0(V) > 0$ for any open set V with $v \in V$. Secondly, suppose that for any $w \neq v$, there is a neighborhood W_w of w such that $\mu_0(W_w) = 0$. Then we have $\mu_0(\mathbb{R}^3 \backslash \{v\}) = 0$, this means that $\mu_0(\{v\}) = 1$, and is contrary to the assumption.

Remark 1.4. Although the results are proved in $\mathbb{R}^3$, the conclusions in this paper still hold for any dimension $N \geq 2$.

2. PROOF OF THEOREM 1.1

Let us first recall two notions about the support and order of measures.

Definition 2.1. Let $\mu \in \mathcal{M}_0$ be a finite positive measure. The support of μ is defined as the smallest closed subset U of $\mathbb{R}^3$ with $\mu(U^c) = 0$.

Definition 2.2. For $\mu, \nu \in \mathcal{M}_0$, we say $\mu \geq \nu$ if for any $A \in \mathcal{B}(\mathbb{R}^3)$ it holds $\mu(A) \geq \nu(A)$, where $\mathcal{B}(\mathbb{R}^3)$ is the family of Borel sets.

Remark 2.3. $\mu \geq \nu$ is equivalent to $\mu(\phi) \geq \nu(\phi)$ for all $\phi \in C_b^+(\mathbb{R}^3)$, positive bounded continuous function.

In the sequel we denote by $B_\varepsilon(\xi)$ the open ball in $\mathbb{R}^3$ with center ξ and radius ε. For $v, w \in \mathbb{R}^3$ and $r > 0$, let $S_{v,w}(r) := \partial B_r((v+w)/2) = \{\xi \in \mathbb{R}^3 : |\xi - (v+w)/2| = r\}$. Simply write $S_{v,w} := S_{v,w}(|v-w|/2)$. For any $r > 0$ and $\omega_0 \in S^2$, put

$$A_{\omega_0}(r) := \{\omega \in S^2 : |\omega - \omega_0| < r\},$$

then

$$\text{the area of surface } A_{\omega_0}(r) = \int_{A_{\omega_0}(r)} d\omega \geq Cr^2, \tag{7}$$

where $C > 0$ is independent of ω_0.

For $\mu \in \mathcal{P}_0$, let $L(\mu)$ be defined by

$$L(\mu)(\xi) := \int_{\mathbb{R}^3} \int_{S^2} \mathcal{B}(\xi - \xi_*, \omega)\mu(d\xi_*)d\omega.$$

Then by (3)

$$\begin{aligned} L(\mu)(\xi) &= 2K_b \int_{\mathbb{R}^3} |\xi - \xi_*|^\beta \mu(d\xi_*) \\ &\leq 2K_b(|\xi|^\beta + M_\beta(\mu)). \end{aligned} \tag{8}$$

Let μ_t be the unique conservative solution to Eq. (5). For $t > s \geq 0$, put

$$G_s^t(\xi) := \exp\left\{-\int_s^t L(\mu_\tau)(\xi)d\tau\right\}.$$

Then by $M_\beta(\mu_\tau) \leq M_2(\mu_\tau) = M_2(\mu_0)$ and (8), we have

$$\begin{aligned} G_s^t(\xi) &\geq \exp\left\{-2K_b(|\xi|^\beta + M_2(\mu_0))(t-s)\right\} \\ &\geq \exp\left\{-C_0(1 + |\xi|^\beta)(t-s)\right\} =: h_s^t(\xi) \end{aligned}$$

for some $C_0 > 0$.

By the uniqueness to Eq. (5), it is easy to see that the following Duhamel formula holds in the dual sense:

$$\mu_t = G_0^t \mu_0 + \int_0^t Q^+(\mu_s) G_s^t ds,$$

where $Q^+(\mu)$ is defined by

$$Q^+(\mu)(\phi) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} [\phi(\xi') + \phi(\xi_*')]\mathcal{B}(\xi - \xi_*, \omega)\mu_s(d\xi)\mu_s(d\xi_*)d\omega.$$

More precisely, for any $\phi \in C_b^+(\mathbb{R}^3)$

$$\mu_t(\phi) = \mu_0(\phi G_0^t) + \int_0^t Q^+(\mu_s)\big(G_s^t \phi\big)ds. \tag{9}$$

Before proving our main result, we first give a useful lemma.

Lemma 2.4. *Let $v, w \in \mathbb{R}^3$. The mapping $S^2 \ni \omega \mapsto w_v(\omega) := v - (v - w, \omega)\omega \in S_{v,w}$ is onto.*

Proof: Noting that

$$|v - w - 2(v - w, \omega)\omega| = |v - w|,$$

we have $w_v \in S_{v,w}$. Let us prove that $\omega \mapsto w_v(\omega)$ is onto. For $\xi \in S_{v,w}$, we obviously have

$$(v - \xi, v - w) = |v - \xi|^2.$$

If $\xi = v$, we may take $\omega \in S^2$ such that $\omega \perp (v - w)$. If $\xi \neq v$, we may take $\omega = \frac{v-\xi}{\sqrt{(v-\xi, v-w)}} \in S^2$ such that $w_v(\omega) = \xi$. □

In the sequel we shall fix the points v, w in Theorem 1.1, and a positive number r_0 being less than $|v - w|/4$.

Without any loss of generality, we may assume that $b(\theta) \geq b_0 > 0$. In fact, from the proof below it is seen that the angle $\theta = \arccos(|(\xi - \xi_*, \omega)|/|\xi - \xi_*|)$ can be taken apart from 0 and $\pi/2$. If we let r_0 and ε_0 below small enough, θ will be near $\pi/4$.

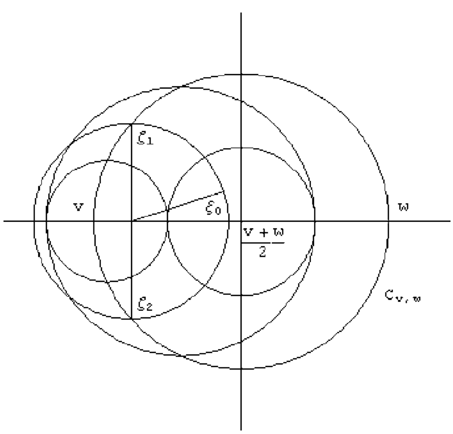
One makes the following convention: the positive constant $C = C(v, w, \beta, b_0, r_0, \varepsilon_0)$ has different values in different occasions.

We first prove the following lemma.

Lemma 2.5. *For any $\xi_0 \in B_{r_0}((v + w)/2)$ and $0 < \varepsilon < r_0/2$, we have for any $t > 0$*

$$\mu_t(B_\varepsilon(\xi_0)) \geq Ct^2 e^{-Ct} \varepsilon^6 f^2(\varepsilon/25).$$

Proof: First of all, we consider the plane through v, w and ξ_0, which intersecting with $S_{v,w}$ gives a large circle $C_{v,w}$(see the following figure).



From the above figure it is clear that there exist $\zeta_1, \zeta_2 \in C_{v,w}$ such that

$$\xi_0 \in S_{\zeta_1,\zeta_2}, \quad |v-w| > |\zeta_1 - v| = |\zeta_2 - v| \geq \frac{|v-w|}{5},$$
$$|\zeta_1 - \zeta_2| > 2|v-w|/5.$$

Moreover, noting that $S^2 \ni \omega \mapsto \zeta_1 - (\zeta_1 - \zeta_2, \omega)\omega \in S_{\zeta_1,\zeta_2}$ is onto, we have for some $\omega_0 \in S^2$

$$\xi_0 = \zeta_1 - (\zeta_1 - \zeta_2, \omega_0)\omega_0.$$

Hence for any $(\xi, \xi_*, \omega) \in B_{\varepsilon/5}(\zeta_1) \times B_{\varepsilon/5}(\zeta_2) \times A_{\omega_0}(\varepsilon/(5|v-w|))$, we have

$$\begin{aligned}
|\xi - (\xi - \xi_*, \omega)\omega - \xi_0| &= |\xi - (\xi - \xi_*, \omega)\omega - \zeta_1 - (\zeta_1 - \zeta_2, \omega_0)\omega_0| \\
&\leq 2|\xi - \zeta_1| + |\xi_* - \zeta_2| + 2|\zeta_1 - \zeta_2| \cdot |\omega - \omega_0| \\
&\leq \varepsilon.
\end{aligned}$$

This means

$$\{\xi - (\xi - \xi_*, \omega)\omega : \xi \in B_{\varepsilon/5}(\zeta_1), \xi_* \in B_{\varepsilon/5}(\zeta_2),$$
$$\omega \in A_{\omega_0}(\varepsilon/(5|v-w|))\} \subset B_\varepsilon(\xi_0).$$

Thus, from (9) and (7) we obtain

$$\begin{aligned}
\mu_t(B_\varepsilon(\xi_0)) &\geq \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \mathcal{B}(\xi - \xi_*, \omega) 1_{B_\varepsilon(\xi_0)}(\xi') G_s^t(\xi') \mu_s(d\xi) \mu_s(d\xi_*) d\omega ds \\
&\geq \int_0^t \int_{B_{\varepsilon/5}(\zeta_1)} \int_{B_{\varepsilon/5}(\zeta_2)} \int_{A_{\omega_0}(\varepsilon/(5|v-w|))} \\
&\quad \times |\xi - \xi_*|^\beta b(\theta) h_s^t(\xi') \mu_s(d\xi) \mu_s(d\xi_*) d\omega ds \\
&\geq C\varepsilon^2 \int_0^t e^{-C(t-s)} \mu_s(B_{\varepsilon/5}(\zeta_1)) \mu_s(B_{\varepsilon/5}(\zeta_2)) ds,
\end{aligned}$$

where in the last step we have used that $|\xi'| \leq 2|\xi| + |\xi_*|$ and

$$h_s^t(\xi') \geq \exp\left\{-C_0(1 + (2|\xi| + |\xi_*|)^\beta)(t-s)\right\}.$$

Let us now estimate $\mu_s(B_{\varepsilon/5}(\zeta_1))$ and $\mu_s(B_{\varepsilon/5}(\zeta_2))$. It is the same reason as above that there is an $\omega_1 \in S^2$ such that $v - (v - w, \omega_1)\omega_1 = \zeta_1$. Thus we have

$$\Big\{\xi - (\xi - \xi_*, \omega)\omega : \xi \in B_{\varepsilon/25}(v), \xi_* \in B_{\varepsilon/25}(w),$$
$$|\omega - \omega_0| < \frac{\varepsilon}{25|v-w|}\Big\} \subset B_{\varepsilon/5}(\zeta_1).$$

Therefore, applying the Duhamel formula (9) again, we obtain

$$\begin{aligned}
&\mu_t(B_{\varepsilon/5}(\zeta_1))\\
&\geq \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \mathcal{B}(\xi-\xi_*,\omega)1_{B_{\varepsilon/5}(\xi_0)}\\
&\quad\times(\xi')G_s^t(\xi')G_0^s(\xi)G_0^s(\xi_*)\mu_0(d\xi)\mu_0(d\xi_*)d\omega ds\\
&\geq \int_0^t \int_{B_{\varepsilon/25}(\zeta_1)}\int_{B_{\varepsilon/25}(\zeta_2)}\int_{A_{\omega_0}(\varepsilon/(25|v-w|))}\\
&\quad\times|\xi-\xi_*|^\beta b(\theta)h_s^t(\xi')h_0^s(\xi)h_0^s(\xi_*)\mu_0(d\xi)\mu_0(d\xi_*)d\omega ds\\
&\geq C\varepsilon^2\left(\int_0^t e^{-C(t-s)}e^{-2Cs}ds\right)\mu_0(B_{\varepsilon/25}(\zeta_1))\mu_0(B_{\varepsilon/25}(\zeta_2))\\
&\geq C\varepsilon^2 f(\varepsilon/25)e^{-Ct}(1-e^{-Ct}).
\end{aligned}$$

Combining the above calculation gives the result. □

The following two lemmas will be used to produce the iteration program as in Ref. 19.

Lemma 2.6. *For any $\xi_0\in\mathbb{R}^3$, denote $r(\xi_0)=|\xi_0-(v+w)/2|$. Then for any $\varepsilon<r(\xi_0)/5$, there exist points $\zeta_1,\zeta_2\in S_{v,w}(\frac{r(\xi_0)}{\sqrt{2}})$ and $\omega_0\in S^2$ such that*

$$|\zeta_1-\zeta_2|=r(\xi_0),\quad |\xi_0-\zeta_1|=|\xi_0-\zeta_2|=\frac{r(\xi_0)}{\sqrt{2}},$$

$$\xi_0=\zeta_1-(\zeta_1-\zeta_2,\omega_0)\omega_0.$$

Therefore,

$$\left\{\xi-(\xi-\xi_*,\omega)\omega : \xi\in B_{\varepsilon/5}(\zeta_1),\xi_*\in B_{\varepsilon/5}(\zeta_2),|\omega-\omega_0|<\frac{\varepsilon}{5r(\xi_0)}\right\}\subset B_\varepsilon(\xi_0).$$

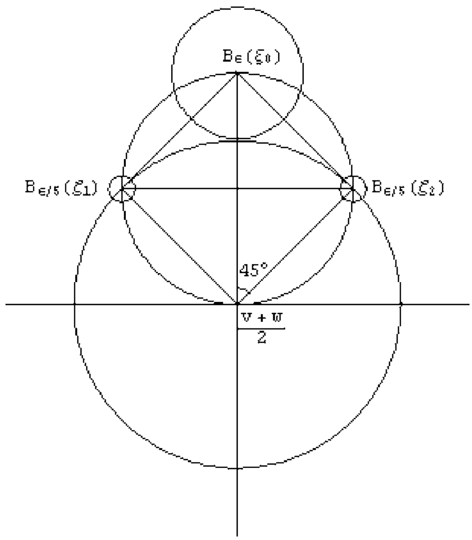
Proof: The existences of ζ_1,ζ_2,ω_0 follow from Lemma 2.4. See the following figure. □

Lemma 2.7. *For any $t>s\geq 0$, it holds that for each $\phi\in C_b^+(\mathbb{R}^3)$*

$$\mu_t(\phi/G_0^t)\geq\mu_s(\phi/G_0^s).$$

In particular,

$$\mu_t\geq G_s^t\mu_s. \tag{10}$$



Proof: A simple calculation gives

$$\frac{d}{dt}\mu_t(\phi/G_0^t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2}[\phi(\xi')/G_0^t(\xi') + \phi(\xi_*')/G_0^t(\xi_*')]\mu_t(d\xi)\mu_t(d\xi_*)d\omega \geq 0.$$

This shows that $t \mapsto \mu_t(\phi/G_0^t)$ is increasing, and the result follows. □

Now we can give

Proof of Theorem 1.1. Let $\xi_0 \in \mathbb{R}^3$ be such that $r(\xi_0) := |\xi_0 - (v+w)/2| \geq r_0$, and $0 < \varepsilon < \varepsilon_0 < 1$ be sufficiently small. For any $t > 0$, by Lemma 2.6. and (9) (10) we have

$$\begin{aligned}
&\mu_t(B_\varepsilon(\xi_0))\\
&\geq \int_{\frac{t}{2}}^t Q^+\big(G^s_{\frac{t}{2}}\mu_{\frac{t}{2}}\big)\big(1_{B_\varepsilon(\xi_0)}G_s^t\big)ds \qquad (11)\\
&= \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \mathcal{B}(\xi-\xi_*,\omega)1_{B_\varepsilon(\xi_0)}\\
&\quad\times(\xi')G_s^t(\xi')G^s_{\frac{t}{2}}(\xi)G^s_{\frac{t}{2}}(\xi_*)\mu_{\frac{t}{2}}(d\xi)\mu_{\frac{t}{2}}(d\xi_*)d\omega ds\\
&\geq b_0\int_{\frac{t}{2}}^t\int_{B_{\varepsilon/5}(\zeta_1)}\int_{B_{\varepsilon/5}(\zeta_2)}\int_{A_{\omega_0}(\varepsilon/(5r(\xi_0)))}
\end{aligned}$$

$$\times |\xi - \xi_*|^\beta h_s^t(\xi') h_{\frac{t}{2}}^s(\xi) h_{\frac{t}{2}}^s(\xi_*) \mu_{\frac{t}{2}}(d\xi)\mu_{\frac{t}{2}}(d\xi_*) d\omega ds$$

$$\geq b_0 \left|\frac{r(\xi_0) - 2\varepsilon/5}{\sqrt{2}}\right|^\beta \frac{\varepsilon^2}{25 \cdot r(\xi_0)^2} \left(\int_{\frac{t}{2}}^t h_s^t(C|\xi_0|) h_{\frac{t}{2}}^s(C|\xi_0|) ds\right)$$

$$\times \mu_{\frac{t}{2}}(B_{\varepsilon/5}(\zeta_1))\mu_{\frac{t}{2}}(B_{\varepsilon/5}(\zeta_2))$$

$$\geq C_3 \varepsilon^2 r(\xi_0)^{\beta-2} t e^{-C_4 t|\xi_0|^\beta} \mu_{\frac{t}{2}}(B_{\varepsilon/5}(\zeta_1))\mu_{\frac{t}{2}}(B_{\varepsilon/5}(\zeta_2)). \tag{12}$$

Here the constant C_3 is less than 1 and independent of ε, t, ξ_0.

Let $n = [\log \frac{r(\xi_0)}{r_0} / \log \sqrt{2}] + 2$, where $[a]$ denotes the integer part of real number a. By $r(\xi_0) \leq |\xi_0| + |v + w|/2$, and noting that

$$2^n = e^{n \log 2} \leq 4 e^{2 \log \frac{2r(\xi_0)}{|v-w|}} \leq \frac{16 r(\xi_0)^2}{|v - w|^2} \leq C(|\xi_0|^2 + 1),$$

we have for any $\delta > 0$

$$n 2^n \leq C|\xi_0|^{2+\delta}. \tag{13}$$

Iterating the inequality (12) $n - 2$ times, we find that there are 2^{n-2} points $\xi_{0i} \in B_{r_0}((v + w)/2)$ such that

$$\mu_t(B_\varepsilon(\xi_0)) \geq (C_3)^{\sum_{i=0}^{n-3} 2^i} \prod_{i=0}^{n-3} \left(\frac{\varepsilon^2 r(\xi_0)^{\beta-2}}{5^{2i}\sqrt{2^i}}\right)^{2^i} \prod_{i=1}^{2^{n-2}} \mu_{t/2^n}(B_{\varepsilon/5^n}(\xi_{0i}))$$

$$\times \exp\left\{-C_0 t \left(|\xi_0|^\beta + \left(\frac{|v + w|}{2}\right)^\beta \sum_{i=0}^{n-3} 2^i + \sum_{i=0}^{n-3} 2^i \left(\frac{r(\xi_0)}{\sqrt{2^i}}\right)^\beta\right)\right\}.$$

Here we have used the fact that for any $\zeta \in S_{v,w}(r(\xi_0)/\sqrt{2^i})$

$$|\zeta|^\beta \leq \left(\frac{|r(\xi_0)|}{\sqrt{2^i}}\right)^\beta + \left(\frac{|v + w|}{2}\right)^\beta.$$

By Lemma 2.5 and (13), a careful calculation yields the desired estimate.

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